TORSION AND THE DIFFERENTIAL FAMILY INDEX THEOREM

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Dedicated to my father Kar-Ming Ho

Abstract. We prove the differential family index theorem by the flat family index theorem and the differential Grothendieck-Riemann-Roch theorem.

Contents

1. Introduction

Given a proper submersion $\pi : X \to B$ between manifolds with closed spin^c fibers of even relative dimension, the differential family index theorem [9, Theorem 7.35] (dFIT for short)

$$\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}} = \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{K}_{\mathrm{FL}}(B) \tag{1}$$

equates the Freed–Lott differential analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}$ [9, Definition 7.27] and the Freed–Lott differential topological index $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}$ [9, Definition 5.34] as homomorphisms on the Freed–Lott differential K-groups. (1) is a refined index theorem as it implies the Atiyah–Singer family index theorem [1]

$$\mathrm{ind}^{\mathrm{a}} = \mathrm{ind}^{\mathrm{t}} : K(X) \to K(B). \tag{2}$$

Apply the Freed–Lott differential Chern character [9, §8.13] to (1)

$$\widehat{\mathrm{ch}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(B) \to \widehat{H}^{\mathrm{even}}(B;\mathbb{R}/\mathbb{Q}), \tag{3}$$

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where $\widehat{H}^{\mathrm{even}}(X;\mathbb{R}/\mathbb{Q})$ is the ring of Cheeger-Simons differential characters [8] with coefficients in $\mathbb{Q}$ of even degree, we obtain the differential Grothendieck–Riemann–Roch theorem [9, Corollary 8.26] (dGRR for short), i.e., the following diagram commutes.

$$\begin{tikzcd} \widehat{K}_{\mathrm{FL}}(X) \arrow[r, "\widehat{\mathrm{ch}}_{\mathrm{FL}}"] \arrow[d, "\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}"'] & \widehat{H}^{\mathrm{even}}(X;\mathbb{R}/\mathbb{Q}) \arrow[d, "\widehat{\int_{X/B}}\widehat{\mathrm{Todd}}(T^VX,\widehat{\nabla}^{T^VX})*(\cdot)"] \\ \widehat{K}_{\mathrm{FL}}(B) \arrow[r, "\widehat{\mathrm{ch}}_{\mathrm{FL}}"'] & \widehat{H}^{\mathrm{even}}(B;\mathbb{R}/\mathbb{Q}) \end{tikzcd} \tag{4}$$

(4) simultaneously implies the Grothendieck–Riemann–Roch theorem, i.e., the following diagram commutes,

$$\begin{tikzcd} K(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "\mathrm{ind}^{\mathrm{a}}"'] & H^{\mathrm{even}}(X;\mathbb{Q}) \arrow[d, "\int_{X/B}\mathrm{Todd}(T(X/B))\cup(\cdot)"] \\ K(B) \arrow[r, "\mathrm{ch}"'] & H^{\mathrm{even}}(B;\mathbb{Q}) \end{tikzcd} \tag{5}$$

and the local family index theorem [3]

$$\mathrm{ch}(\nabla^{\mathrm{ind}(\mathsf{D}^E)}) = \int_{X/B} \mathrm{Todd}(\widehat{\nabla}^{T^VX}) \wedge \mathrm{ch}(\nabla^E) - d\widetilde{\eta}, \tag{6}$$

where $\widetilde{\eta}$ is the Bismut–Cheeger eta form [4].

Bunke–Schick's differential K-group $\widehat{K}_{\mathrm{BS}}$ [6] is another equivalent model of differential K-theory, in the sense that $\widehat{K}_{\mathrm{BS}}$ is isomorphic to $\widehat{K}_{\mathrm{FL}}$ by a unique natural isomorphism [7, Corollary 4.4]. Bunke–Schick also define a differential analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{BS}}$ and a differential Chern character $\widehat{\mathrm{ch}}_{\mathrm{BS}}$, and prove the dGRR in their setting [6, Theorem 6.19]. Since $\mathrm{ind}^{\mathrm{a}}_{\mathrm{BS}}$ is essentially equivalent to $\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}$ [6, 5.3.5] and the differential Chern character is unique [6, Proposition 6.3], the dGRRs in these settings are essentially equivalent. For a proof of the dGRR in the setting of $\widehat{K}_{\mathrm{FL}}$ which does not make use of the dFIT, see [10].

In [13] J. Lott constructs a geometric model of $\mathbb{R}/\mathbb{Z}$ K–theory $K^{-1}_{\mathrm{L}}(X;\mathbb{R}/\mathbb{Z})$, which is closely related to Karoubi's multiplicative K-theory [12], and proves a family index theorem [13, Corollary 3]

$$\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}} = \mathrm{ind}^{\mathrm{t}}_{\mathrm{L}} : K^{-1}_{\mathrm{L}}(X;\mathbb{R}/\mathbb{Z}) \to K^{-1}_{\mathrm{L}}(B;\mathbb{R}/\mathbb{Z}), \tag{7}$$

called the $\mathbb{R}/\mathbb{Z}$ family index theorem. Apply the $\mathbb{R}/\mathbb{Z}$ Chern character

$$\mathrm{ch}_{\mathbb{R}/\mathbb{Q}} : K^{-1}_{\mathrm{L}}(B;\mathbb{R}/\mathbb{Z}) \to H^{\mathrm{odd}}(B;\mathbb{R}/\mathbb{Q})$$

to (7) we obtain the $\mathbb{R}/\mathbb{Z}$ Grothendieck–Riemann–Roch theorem ($\mathbb{R}/\mathbb{Z}$ GRR for short) [13, Corollary 4], i.e., the following diagram commutes.

$$\begin{array}{ccc} K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z}) & \xrightarrow{\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}} & H^{\mathrm{odd}}(B;\mathbb{R}/\mathbb{Q}) \\ {\scriptstyle \mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \int_{X/B}\mathrm{Todd}(X/B)\cup(\cdot)} \\ K_{\mathrm{L}}^{-1}(B;\mathbb{R}/\mathbb{Z}) & \xrightarrow[\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}]{} & H^{\mathrm{odd}}(B;\mathbb{R}/\mathbb{Q}) \end{array} \tag{8}$$

$\mathbb{R}/\mathbb{Z}$ K-theory is now known as the flat part of differential K-theory, i.e., there exists a canonical inclusion $i : K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z}) \hookrightarrow \widehat{K}_{\mathrm{FL}}(X)$ such that elements in $K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$ has zero "curvature" when regarded in $\widehat{K}_{\mathrm{FL}}(X)$. Henceforth we replace the term "$\mathbb{R}/\mathbb{Z}$" by "flat". Moreover, the flat indexes $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}$ and $\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}$ can be considered as special cases of the differential indexes $\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}$ and $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}$ [9, Proposition 7.37, Proposition 8.10]. Thus (7) can be regarded as a special case of (1) and similarly (8) can be regarded as a special case of (4).

In this paper we show that the flat FIT (7) and the condensed proof of dGRR [10, Theorem 1] imply the dFIT (1). We prove the dFIT by considering the cases where an element in $\widehat{K}_{\mathrm{FL}}(X)$ is torsion or torsion-free separately. If $\mathcal{E} \in \widehat{K}_{\mathrm{FL}}(X)$ is torsion-free then both $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E})$ and $\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E})$ are torsion-free by the (condensed proof of the) dGRR and [9, Proposition 8.19]. Thus proving the dFIT in the torsion-free case is equivalent to proving $\widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E})) = \widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E}))$ as the Freed-Lott differential Chern character $\widehat{\mathrm{ch}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(\cdot) \to \widehat{H}^{\mathrm{even}}(\cdot;\mathbb{R}/\mathbb{Q})$ is a rational isomorphism. If $\mathcal{E} - [n] \in \widehat{K}_{\mathrm{FL}}(X)$ is torsion, we prove that it is induced by an element, denoted by $\mathcal{E}' - [n]'$, in $K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$. By [9, Proposition 7.39] it follows that $\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E} - [n]) = i(\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}' - [n]')$. On the other hand, we prove that $\mathrm{ind}^{\mathrm{a}}_{\mathrm{t}}(\mathcal{E} - [n]) = i(\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}(\mathcal{E}' - [n]')$ (Proposition 2). Thus in the torsion case the dFIT reduces to the flat FIT. Note that Proposition 2 is proved in [9, Proposition 8.10], but that is a consequence of the flat FIT and the dFIT. The proof of Proposition 2 does not make use of the flat FIT and the dFIT.

The paper is organized as follows. Section 2 reviews Freed–Lott differential K-theory, Lott flat K-theory and the pairing between flat K-theory and odd K-homology. The main results are proved in Section 3.

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2. Background materials

2.1. **Freed–Lott differential K-theory.** Let X be a compact smooth manifold. The Freed–Lott differential K-group $\widehat{K}_{\mathrm{FL}}(X)$ is the abelian group

generated by quadruples $\mathcal{E} = (E, h, \nabla, \phi)$, where $(E, h, \nabla) \to X$ is a complex vector bundle with a hermitian metric h and a unitary connection ∇, and $\phi \in \dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$. The only relation is $\mathcal{E}_1 = \mathcal{E}_2$ if and only if there exists a generator $(F, h^F, \nabla^F, \phi^F)$ of $\widehat{K}_{\mathrm{FL}}(X)$ such that $E_1 \oplus F \cong E_2 \oplus F$ and $\phi_2 - \phi_1 = \mathrm{CS}(\nabla^{E_1} \oplus \nabla^F, \nabla^{E_2} \oplus \nabla^F)$, where $\mathrm{CS}(\nabla^1, \nabla^0) \in \dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$ is the Chern-Simons form of two connections ∇^1, ∇^0 on a complex vector bundle $E \to X$ such that

$$\mathrm{ch}(\nabla^1) - \mathrm{ch}(\nabla^0) = d\,\mathrm{CS}(\nabla^1, \nabla^0).$$

Note that every element $\mathcal{E} - \mathcal{F} \in \widehat{K}_{\mathrm{FL}}(X)$ can be written in the form

$$\mathcal{E}' - [n]. \tag{9}$$

Here $\mathcal{E}' = (E \oplus G, h^E \oplus h^G, \nabla^E \oplus \nabla^G, \phi^E + \phi^G)$, where $(G, h^G, \nabla^G, \phi^G)$ a generator of $\widehat{K}_{\mathrm{FL}}(X)$ such that

$$(F \oplus G, h^F \oplus h^G, \nabla^F \oplus \nabla^G, \phi^F + \phi^G) = (\mathbb{C}^n, h, d, 0) =: [n].$$

The existence of the connection ∇^G such that $\mathrm{CS}(\nabla^F \oplus \nabla^G, d) = 0$, where d is the trivial connection on the trivial bundle $X \times \mathbb{C}^n \to X$, follows from [14, Theorem 1.15]. Here $\phi^G := -\phi^F$. Henceforth we assume an element of $\widehat{K}_{\mathrm{FL}}(X)$ is of the form $\mathcal{E} - [n]$.

In the following hexagon, the diagonal sequences are exact, and every triangle and square commutes [9]:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr, "i"] & & K(X;\mathbb{Z}) \arrow[ur] \arrow[dr, "\mathrm{ch}_{\mathbb{R}}"] & \\
H^{\mathrm{odd}}(X;\mathbb{R}) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \widehat{K}_{\mathrm{FL}}(X) \arrow[ur, "\delta"] \arrow[dr, "\mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}"] & & H^{\mathrm{even}}(X;\mathbb{R}) \\
& \frac{\Omega^{\mathrm{odd}}(X)}{\Omega^{\mathrm{odd}}_{\mathrm{BU}}(X)} \arrow[rr, "d"'] \arrow[ur, "j"] & & \Omega^{\mathrm{even}}_{\mathrm{BU}}(X) \arrow[ur, "\mathrm{dr}"] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd} \tag{10}$$

where $\mathrm{ch}_{\mathbb{R}} := r \circ \mathrm{ch} : K(X) \to H^{\mathrm{even}}(X; \mathbb{R})$,

$$\Omega^\bullet_{\mathrm{BU}}(X) := \{\omega \in \Omega^\bullet_{d=0}(X) | [\omega] \in \mathrm{Im}(\mathrm{ch}^\bullet : K^{-(\bullet \mod 2)} \to H^\bullet(X; \mathbb{Q}))\},$$

where $\bullet \in \{\mathrm{even}, \mathrm{odd}\}$. The maps are defined as follows:

$$\delta(\mathcal{E}) = [E],\ \mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathcal{E}) = \mathrm{ch}(\nabla) + d\phi,\ j(\phi) = (0, 0, d, \phi),$$

i is the natural inclusion map, dr is the de Rham map, and the sequence of maps $(\alpha, \beta, \mathrm{ch}_{\mathbb{R}})$ can be regarded as the Bockstein sequence in K-theory as we may identify $H^{\bullet}(X;\mathbb{R})$ with $K^{\bullet}(X;\mathbb{R}) := K^{\bullet}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ via the Chern character.

The Freed–Lott differential Chern character

$$\widehat{\mathrm{ch}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{H}^{\mathrm{even}}(X;\mathbb{R}/\mathbb{Q})$$

is defined by

$$\widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathcal{E}) = \widehat{\mathrm{ch}}(E, h, \nabla) + i_2(\phi),$$

where $\widehat{\mathrm{ch}}(E, h, \nabla)$ is the unique natural differential Chern character defined in [8, §4] such that its form part is $\mathrm{ch}(\nabla)$ and its cohomology part is $\mathrm{ch}(E)$. The map $i_2 : \dfrac{\Omega^{\mathrm{odd}}(X)}{\Omega^{\mathrm{odd}}_{\mathbb{Q}}(X)} \to \widehat{H}^{\mathrm{even}}(X;\mathbb{R}/\mathbb{Q})$ is the injective map defined in [8, Theorem 1.1]. Henceforth we suppress the metric h and just write $\widehat{\mathrm{ch}}(E, \nabla)$.

Let $\pi : X \to B$ be a proper submersion with closed spin^c fibers of even relative dimension. Given the geometric data in [9, §3.1], the Freed–Lott differential analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{K}_{\mathrm{FL}}(B)$ [9, Definition 3.11] is defined by

$$\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E}) = \left(\ker(\mathsf{D}^E), h^{\ker(\mathsf{D}^E)}, \nabla^{\ker(\mathsf{D}^E)}, \int_{X/B} \mathrm{Todd}(\widehat{\nabla}^{T^V X}) \wedge \phi + \widetilde{\eta}(\mathcal{E}) \right), \tag{11}$$

where D^E is the family of Dirac operators on the twisted spinor bundle $\mathcal{S}^V(X) \otimes E \to X$. Here $\widehat{\nabla}^{T^V X} := \nabla^{T^V X} \otimes \nabla^{L(X)}$, where $\nabla^{T^V X}$ is the unique lift of the projection of the Levi-Civita connection on $TX \to X$ onto the vertical bundle $T^V X \to X$ to the spinor bundle, and $\nabla^{L(X)}$ is a unitary connection on the characteristic line bundle $L(X) \to X$ associated to the spin^c structure. $\ker(\mathsf{D}^E) \to B$ is assumed to form a superbundle, and $\widetilde{\eta}(\mathcal{E})$ is the Bismut-Cheeger eta form [4], satisfying

$$d\widetilde{\eta}(\mathcal{E}) = \int_{X/B} \mathrm{Todd}(\widehat{\nabla}^{T^V X}) \wedge \mathrm{ch}(\nabla^E) - \mathrm{ch}(\nabla^{\ker(\mathsf{D}^E)}).$$

For the construction of $\ker(\mathsf{D}^E) \to B$, $h^{\ker(\mathsf{D}^E)}$ and $\nabla^{\ker(\mathsf{D}^E)}$, see [2].

There are similar constructions of $\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E})$ [9, Definition 7.27] and $\widetilde{\eta}(\mathcal{E})$ [9, (7.25)] when the family of kernels $\ker(\mathsf{D}^E)$ do not form a superbundle over B.

The differential topological index $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{K}_{\mathrm{FL}}(B)$ is defined to be the composition of a modified embedding pushforward and a projection pushforward. Roughly speaking, the modified embedding pushforward is the geometric construction of the embedding pushforward given in [5] (see also [9, §4.1]) with a correction term (see [9, (5.6)]).

We briefly recall that construction of the embedding pushforward, and refer to [9, §4, 5] for the details. Let $\mathcal{E} \in \widehat{K}_{\mathrm{FL}}(X)$ and $\iota : X \hookrightarrow Y$ a proper embedding of manifolds and assume the codimension of X in Y is even. As in

[9, §5] we assume for $b \in B$, the map $\iota_b : X_b \to Y_b$ is an isometric embedding. The embedding pushforward $\widehat{\iota}_* : \widehat{K}_{\mathrm{FL}}(X) \to {}_\delta\widehat{K}_{\mathrm{FL}}(Y)$ [9, Definition 4.14] is defined to be

$$\widehat{\iota}_*(\mathcal{E}) = \Big(F, h^F, \nabla^F, \frac{\phi^E}{\mathrm{Todd}(\widehat{\nabla}^\nu)} \wedge \delta_X - \gamma\Big).$$

Here ${}_\delta\widehat{K}_{\mathrm{FL}}(Y)$ is the currential K-theory [9, §2.28], $\nu \to X$ is the normal bundle of X in Y and is assumed to be (differential) spin^c, $\widehat{\nabla}^\nu$ is the induced connection on the spinor bundle $\mathcal{S}(\nu) \to X$, δ_X is the current of integration of X and γ is the Bismut-Zhang current [9, Definition 1.3]. (F, h^F, ∇^F) is a Hermitian bundle with a Hermitian metric h^F and a unitary connection chosen in [9, Lemma 4.4].

Now let $X \to B$ and $Y \to B$ be fiber bundles with X compact. Since the horizontal distributions of these fiber bundles need not be compatible, an odd form $\widetilde{C} \in \dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$ is defined to correct this noncompatibility. $\widetilde{C}$ satisfies

$$d\widetilde{C} = \iota^* \mathrm{Todd}(\widehat{\nabla}^{T^VY}) - \mathrm{Todd}(\widehat{\nabla}^{T^VX}) \wedge \mathrm{Todd}(\widehat{\nabla}^\nu).$$

The modified embedding pushforward $\widehat{\iota}_*^{\mathrm{mod}} : \widehat{K}_{\mathrm{FL}}(X) \to {}_{\mathrm{WF}}\widehat{K}_{\mathrm{FL}}(Y)$ [9, Definition 5.8] is defined to be

$$\widehat{\iota}_*^{\mathrm{mod}}(\mathcal{E}) := \widehat{\iota}_*(\mathcal{E}) - j\bigg(\frac{\widetilde{C}}{\iota^* \mathrm{Todd}(\widehat{\nabla}^{T^VY}) \wedge \mathrm{Todd}(\widehat{\nabla}^\nu)} \wedge \mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathcal{E}) \wedge \delta_X\bigg), \quad (12)$$

where $\mathcal{E} = (E, h^E, \nabla^E, \phi^E)$. See [9, §3.1] for the definition of ${}_{\mathrm{WF}}\widehat{K}_{\mathrm{FL}}(X)$. In particular, if $T^HY|_X \cong T^HX$, by [9, Lemma 5.7] we have $\iota^*\widehat{\nabla}^{T^VY} = \widehat{\nabla}^{T^VX} \oplus \widehat{\nabla}^\nu$ and $\widetilde{C} = 0$. Thus in this case $\widehat{\iota}_*^{\mathrm{mod}} = \widehat{\iota}_*$.

The differential topological index $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{K}_{\mathrm{FL}}(B)$ [9, Definition 5.34] is defined by taking $Y = \mathbb{S}^N \times B$ for some even N and composed with the submersion pushforward $\widehat{\pi}_*^{\mathrm{prod}}$ defined in [9, Lemma 5.13], i.e., $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}} := \widehat{\pi}_*^{\mathrm{prod}} \circ \widehat{\iota}_*^{\mathrm{mod}}$.

The main result of [9] is the dFIT.

Theorem 1. [9, Theorem 7.32] *Let $\pi : X \to B$ be a proper submersion with closed* spin^c *fibers of even relative dimension. Then*

$$\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}} = \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{K}_{\mathrm{FL}}(B).$$

The following proposition gives the differential Chern character of the differential topological index.

Proposition 1. [9, Proposition 8.16] *Let $\pi : X \to B$ be a proper submersion with closed* spin^c *fibers of even relative dimension. Then for $\mathcal{E} \in \widehat{K}_{\mathrm{FL}}(X)$,*

$$\widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E})) = \widehat{\int_{X/B}} \widehat{\mathrm{Todd}}(T^VX, \widehat{\nabla}^{T^VX}) * \widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathcal{E}).$$

Here $\widehat{\int_{X/B}}$ and $*$ is the pushforward [11, §3.4] and the multiplication [8] of differential characters, and $\widehat{\mathrm{Todd}}(T^V X, \widehat{\nabla}^{T^V X})$ is the unique natural differential character defined similarly as $\widehat{\mathrm{ch}}$.

The dGRR follows from $\widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E})) = \widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E}))$ and Proposition 1.

Theorem 2. [9, Corollary 8.23] *Let $\pi : X \to B$ be a proper submersion with closed* spin^c *fibers of even relative dimension. Then the following diagram commutes.*

$$\begin{tikzcd} \widehat{K}_{\mathrm{FL}}(X) \arrow[r, "\widehat{\mathrm{ch}}_{\mathrm{FL}}"] \arrow[d, "\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}"'] & \widehat{H}^{\mathrm{even}}(X; \mathbb{R}/\mathbb{Q}) \arrow[d, "\widehat{\int_{X/B}} \widehat{\mathrm{Todd}}(T^V X, \widehat{\nabla}^{T^V X}) * (\cdot)"] \\ \widehat{K}_{\mathrm{FL}}(B) \arrow[r, "\widehat{\mathrm{ch}}_{\mathrm{FL}}"'] & \widehat{H}^{\mathrm{even}}(B; \mathbb{R}/\mathbb{Q}) \end{tikzcd}$$

i.e., for $\mathcal{E} \in \widehat{K}_{\mathrm{FL}}(X)$, we have

$$\widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E})) = \widehat{\int_{X/B}} \widehat{\mathrm{Todd}}(T^V X, \widehat{\nabla}^{T^V X}) * \widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathcal{E}).$$

For a proof of dGRR which does not make use of Theorem 1, see [10, Theorem 1].

2.2. **The flat K-theory $K^{-1}_{\mathrm{L}}(X; \mathbb{R}/\mathbb{Z})$.** The flat K-group $K^{-1}_{\mathrm{L}}(X; \mathbb{R}/\mathbb{Z})$ is given by generators and relations. A generator $\mathcal{E} = (E, h, \nabla, \phi)$ consists of a Hermitian bundle $E \to X$ with a Hermitian metric h, a unitary connection ∇ and $\phi \in \dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$ such that $\mathrm{ch}(\nabla) - \mathrm{rank}(E) = -d\phi$. One of the relations is the same as the one of $\widehat{K}_{\mathrm{FL}}(X)$, and the other is that elements in $K^{-1}_{\mathrm{L}}(X; \mathbb{R}/\mathbb{Z})$ have virtual rank zero.

By a $\mathbb{Z}_2$-graded element $\mathcal{E} = (E, h^E, \nabla^E, \phi) \in K^{-1}_{\mathrm{L}}(X; \mathbb{R}/\mathbb{Z})$ we mean $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded complex vector bundle over X, $h^E = h^+ \oplus h^-$ and $\nabla^E = \nabla^+ \oplus \nabla^-$, where $h^\pm$ is a Hermitian metric and $\nabla^\pm$ is a unitary connection on $E^\pm \to X$, and $\mathrm{ch}(\nabla^E) := \mathrm{ch}(\nabla^+) - \mathrm{ch}(\nabla^-) = -d\phi$.

One can associate an element $\mathcal{E} - \mathcal{F} \in K^{-1}(X; \mathbb{R}/\mathbb{Z})$ a $\mathbb{Z}_2$-graded element and vice versa [13, p.8].

In [13, Definition 9] the flat Chern character

$$\mathrm{ch}_{\mathbb{R}/\mathbb{Q}} : K^{-1}_{\mathrm{L}}(X; \mathbb{R}/\mathbb{Z}) \to H^{\mathrm{odd}}(X; \mathbb{R}/\mathbb{Q})$$

is defined. Note that $\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}$ is a rational isomorphism.

The flat analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}} : K^{-1}_{\mathrm{L}}(X; \mathbb{R}/\mathbb{Z}) \to K^{-1}_{\mathrm{L}}(B; \mathbb{R}/\mathbb{Z})$ of a given proper submersion $\pi : X \to B$ with closed spin^c fibers of even relative dimension is defined exactly as in (11), i.e. for a $\mathbb{Z}_2$-graded element $\mathcal{E} \in$

$K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$,

$$\mathrm{ind}_{\mathrm{L}}^{\mathrm{a}}(\mathcal{E}) := \left(\ker(\mathsf{D}^E)^{\pm}, h^{\ker(\mathsf{D}^E)^{\pm}}, \nabla^{\ker(\mathsf{D}^E)^{\pm}}, \int_{X/B}\mathrm{Todd}(\widehat{\nabla}^{T^VX})\wedge\phi+\widetilde{\eta}(\mathcal{E})\right),$$

where $\ker(\mathsf{D}^E)^{\pm} = \ker(\mathsf{D}^{E^+})^{\pm}\oplus\ker(\mathsf{D}^{E^-})^{\mp}$, and $\widetilde{\eta}(\mathcal{E}) = \widetilde{\eta}(\mathcal{E}^+) - \widetilde{\eta}(\mathcal{E}^-)$.

The flat analytic index and the flat topological index of elements of the form $\mathcal{E}-\mathcal{F}$ are defined to be the ones of the associated $\mathbb{Z}_2$-graded element.

Theorem 3. [13, Corollary 3] *Let $\pi : X \to B$ be a proper submersion with closed* spin^c *fibers of even relative dimension. Then*

$$\mathrm{ind}_{\mathrm{L}}^{\mathrm{a}} = \mathrm{ind}_{\mathrm{L}}^{\mathrm{t}} : K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z}) \to K_{\mathrm{L}}^{-1}(B;\mathbb{R}/\mathbb{Z}).$$

For a $\mathbb{Z}_2$-graded element $\mathcal{E}\in K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$, we have

$$\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathrm{ind}_{\mathrm{L}}^{\mathrm{t}}(\mathcal{E})) = \int_{X/B}\mathrm{Todd}(X/B)\cup\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E}). \tag{13}$$

Note that $\widehat{\mathrm{ch}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(X)\otimes\mathbb{Q}\to\widehat{H}^{\mathrm{even}}(X;\mathbb{R}/\mathbb{Q})$ is a ring isomorphism, which can be proved by applying the Five lemma to the following diagram

$$\begin{tikzcd}
0 \arrow[r] & K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})\otimes\mathbb{Q} \arrow[r, "i"] \arrow[d, "\mathrm{ch}_{\mathbb{R}/\mathbb{Q}}"', "\cong"] & \widehat{K}_{\mathrm{FL}}(X)\otimes\mathbb{Q} \arrow[r, "\delta"] \arrow[d, "\widehat{\mathrm{ch}}_{\mathrm{FL}}"] & K(X)\otimes\mathbb{Q} \arrow[r] \arrow[d, "\mathrm{ch}"', "\cong"] & 0 \\
0 \arrow[r] & H^{\mathrm{odd}}(X;\mathbb{R}/\mathbb{Q}) \arrow[r, "i_1"'] & \widehat{H}^{\mathrm{even}}(X;\mathbb{R}/\mathbb{Q}) \arrow[r, "\delta_2"'] & H^{\mathrm{even}}(X;\mathbb{Q}) \arrow[r] & 0
\end{tikzcd}$$

where the upper exact sequence is given in (10) and the bottom exact sequence is the Bockstein sequence.

2.3. **Pairing between flat K-theory and K-homology.** Let X be a closed odd-dimensional spin^c manifold. Let $L(X)\to X$ be the characteristic line bundle of the spin^c structure. Let $\mathcal{E} = (E, h^E, \nabla^E, \phi)\in {}_{\delta}\widehat{K}_{\mathrm{FL}}(X)$, and $\mathsf{D}^{X,E}$ be the Dirac-type operator on the spinor bundle $\mathcal{S}(X)\to X$. Recall that the reduced eta-invariant $\bar{\eta}(\mathsf{D}^{X,E})\in\mathbb{R}/\mathbb{Z}$ is defined to be

$$\bar{\eta}(\mathsf{D}^{X,E}) := \frac{1}{2}(\eta(\mathsf{D}^{X,E}) + \dim(\ker(\mathsf{D}^{X,E})) \mod \mathbb{Z}.$$

In [9, Definition 2.33] a modified reduced eta-invariant $\bar{\eta}(X,\mathcal{E})\in\mathbb{R}/\mathbb{Z}$ is defined to be

$$\bar{\eta}(X,\mathcal{E}) := \bar{\eta}(\mathsf{D}^{X,E}) + \int_X\mathrm{Todd}(\widehat{\nabla}^{TX})\wedge\phi \mod \mathbb{Z}.$$

Here $\widehat{\nabla}^{TX} = \nabla^{TX}\otimes\nabla^{L(X)}$, where ∇^{TX} is the unique lift of the Levi-Civita connection on $TX\to X$ to the spinor bundle and $\nabla^{L(X)}$ is a unitary connection on the characteristic line bundle $L(X)\to X$ associated to the spin^c structure.

It is proved in [9, Proposition 2.25] that $\bar{\eta} : {}_{\delta}\widehat{K}_{\mathrm{FL}}(X)\to\mathbb{R}/\mathbb{Z}$ is a well defined homomorphism and

$$\bar{\eta}(X, i(\mathcal{E})) = \langle[X],\mathcal{E}\rangle, \tag{14}$$

where $\mathcal{E}$ is a generator of $K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$ and $[X] \in K_{-1}(X)$ is the fundamental K-homology class. Here $\langle [X], \mathcal{E}\rangle$ is the pairing

$$K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z}) \times K_{-1}(X) \to \mathbb{R}/\mathbb{Z} \tag{15}$$

[13, Proposition 3]. More precisely, by the universal coefficient theorem of [15, (3.1)] we have the following short exact sequence

$$0 \longrightarrow \mathrm{Ext}(K_{-2}(X),\mathbb{R}/\mathbb{Z}) \longrightarrow K^{-1}(X;\mathbb{R}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(K_{-1}(X),\mathbb{R}/\mathbb{Z}) \longrightarrow 0$$

Since $\mathbb{R}/\mathbb{Z}$ is divisible, $\mathrm{Ext}(K_{-2}(X),\mathbb{R}/\mathbb{Z}) = 0$ and therefore $K^{-1}(X;\mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}(K_{-1}(X),\mathbb{R}/\mathbb{Z})$. (14) gives the pairing (15) in analytic terms. $\bar{\eta}(X,\mathcal{E})$ plays an important role in the Freed-Lott's proof of the dFIT.

3. Main results

In this section we prove the main results in this paper.

Let $\mathcal{E} - [n] \in \widehat{K}_{\mathrm{FL}}(X)$ be a torsion element. Then there exists $k \in \mathbb{N}$ such that $k(\mathcal{E} - [n]) = 0 \in \widehat{K}_{\mathrm{FL}}(X)$, which implies the existence of a generator $\mathcal{G} = (G, h^G, \nabla^G, \phi^G)$ of $\widehat{K}_{\mathrm{FL}}(X)$ such that

$$\begin{aligned} kE \oplus G &\cong \mathbb{C}^{kn} \oplus G, \\ -k\phi^E &= \mathrm{CS}(k\nabla^E \oplus \nabla^G, k\nabla^{\mathrm{flat}} \oplus \nabla^G). \end{aligned} \tag{16}$$

These equalities imply that

$$\mathrm{rank}(E) = n \text{ and } k\,\mathrm{ch}(\nabla^E) - k\,\mathrm{rank}(E) = -kd\phi^E. \tag{17}$$

By the definition of $K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$ and (17), $\mathcal{E}$ and $[n]$ are generators of $K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$ and $\mathcal{E} - [n] \in K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$. Denote by $\mathcal{E}'$ and $[n]'$ when $\mathcal{E}$ and $[n]$ are considered as generators of $K^{-1}(X;\mathbb{R}/\mathbb{Z})$. Note that

$$i(\mathcal{E}' - [n]') = \mathcal{E} - [n], \tag{18}$$

where i is given in (10). By [9, Proposition 7.37] we have

$$\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E} - [n]) = \mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(i(\mathcal{E}' - [n]')) = i(\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}' - [n]')), \tag{19}$$

The proof of the following Proposition is inspired by the calculations involved in the proofs of the flat FIT and the dFIT.

Proposition 2. *Let $\mathcal{E} - [n] \in \widehat{K}(X)$ be a torsion element, and $\mathcal{E}' - [n]' \in K^{-1}(X;\mathbb{R}/\mathbb{Z})$ as above. Then $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n]) = i(\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}(\mathcal{E}' - [n]'))$.*

Proof. Since $\mathcal{E} - [n] \in \widehat{K}_{\mathrm{FL}}(X)$ is torsion, let $k \in \mathbb{N}$ and $\mathcal{G} = (G, h^G, \nabla^G, \phi^G)$ a generator of $\widehat{K}_{\mathrm{FL}}(X)$ as in (16) and $i(\mathcal{E}' - [n]') = \mathcal{E} - [n]$ as in (18). Consider the difference

$$h := \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n]) - i(\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}(\mathcal{E}' - [n]')).$$

We prove that $h = 0$. By [9, Lemma 5.36] and the fact that $\mathrm{ch}_{\widehat{K}_{\mathrm{FL}}} \circ i = 0$, we have

$$\begin{aligned}&\mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n])) - \mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(i(\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}(\mathcal{E}' - [n]'))) \\ &= \mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n])) \\ &= k \int_{X/B} \mathrm{Todd}(\widehat{\nabla}^{T^V X}) \wedge (\mathrm{ch}(\nabla^E) - \mathrm{rank}(E) + d\phi^E) \\ &= 0.\end{aligned}$$

By the exactness of one of the diagonal sequences in (10), there exists an element $a \in K^{-1}(B; \mathbb{R}/\mathbb{Z})$ such that $i(a) = h$. To prove $a = 0 \in K^{-1}_{\mathrm{L}}(B; \mathbb{R}/\mathbb{Z})$, it suffices to show that for all $\alpha \in K_{-1}(B; \mathbb{Z})$,

$$\langle \alpha, a \rangle = 0 \in \mathbb{R}/\mathbb{Z}. \tag{20}$$

As proceeded in the proof of [9, Theorem 6.2], we may, without loss of generality, let $\alpha = f_*[M]$ for some smooth map $f : M \to B$, where M is a closed odd-dimensional spin^c manifold, and $[M]$ is the fundamental K-homology in $K_{-1}(M)$. Since $\langle \alpha, a \rangle = \langle [M], f^* a \rangle$, we pull everything back to M and we may assume B is an arbitrary closed odd-dimensional spin^c manifold. Thus proving (20) is equivalent to proving

$$\langle [B], a \rangle = 0 \in \mathbb{R}/\mathbb{Z}. \tag{21}$$

Since

$$\langle [B], a \rangle = \bar{\eta}(B, \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n])) - \bar{\eta}(B, i(\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}(\mathcal{E}' - [n]'))) \mod \mathbb{Z},$$

proving (21) is equivalent to proving

$$\bar{\eta}(B, \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n])) = \bar{\eta}(B, i(\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}(\mathcal{E}' - [n]'))) \mod \mathbb{Z}. \tag{22}$$

In the following, we write $a \equiv b$ as $a = b \mod \mathbb{Z}$. By [9, (6.7)], we have

$$\begin{aligned}\bar{\eta}(B, \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n])) &\equiv \bar{\eta}(\mathsf{D}^{X, E-n}) + \int_X \frac{\iota^* \mathrm{Todd}(\widehat{\nabla}^{T^V(\mathbb{S}^N \times B)})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi^E \\ &\quad - \int_X \frac{\pi^* \mathrm{Todd}(\widehat{\nabla}^{TB})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge \mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathcal{E} - [n]) \qquad (23) \\ &\equiv \bar{\eta}(\mathsf{D}^{X, E-n}) + \int_X \frac{\iota^* \mathrm{Todd}(\widehat{\nabla}^{T^V(\mathbb{S}^N \times B)})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi^E\end{aligned}$$

as $\mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathcal{E} - [n]) = 0$ by (17). On the other hand, by [13, (49)], we have

$$\begin{aligned}\bar{\eta}(B, i(\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}(\mathcal{E}' - [n]'))) &\equiv \langle [B], \mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}(\mathcal{E}' - [n]') \rangle \\ &= \langle \pi^*[B], \mathcal{E} - [n] \rangle \\ &= \langle [X], \mathcal{E} - [n] \rangle \\ &= \bar{\eta}(X, \mathcal{E} - [n]) \\ &\equiv \bar{\eta}(\mathsf{D}^{X, E-n}) + \int_X \mathrm{Todd}(\widehat{\nabla}^{TB}) \wedge \phi^E.\end{aligned} \tag{24}$$

From (23) and (24) we have

$$\begin{aligned}
&\bar\eta(B, \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E}-[n])) - \bar\eta(B, i(\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}(\mathcal{E}'-[n]'))) \\
&\equiv \int_X \left(\frac{\iota^* \operatorname{Todd}(\widehat\nabla^{T^V(\mathbb{S}^N\times B)})}{\operatorname{Todd}(\widehat\nabla^{\nu})} - \operatorname{Todd}(\widehat\nabla^{TB}) \right) \wedge \phi^E \\
&\equiv \int_X \left(\frac{\iota^* \operatorname{Todd}(\widehat\nabla^{T^V(\mathbb{S}^N\times B)}) - \operatorname{Todd}(\widehat\nabla^{TB}) \wedge \operatorname{Todd}(\widehat\nabla^{\nu})}{\operatorname{Todd}(\widehat\nabla^{\nu})} \right) \wedge \phi^E.
\end{aligned} \tag{25}$$

Since $\mathrm{ch}_{\widehat K_{\mathrm{FL}}}(\mathcal{E}-[n]) = 0$, it follows from (12) that

$$\widehat\iota^{\mathrm{mod}}_*(\mathcal{E}-[n]) = \widehat\iota_*(\mathcal{E}-[n]).$$

Since the purpose of the modified embedding pushforward $\widehat\iota^{\mathrm{mod}}_*$ is to correct the noncompatibility of the horizontal distributions $T^H(\mathbb{S}^N\times B)$ and $T^H X$, and in our case the modified embedding pushforward equals the embedding pushforward, we may assume horizontal distributions $T^H(\mathbb{S}^N\times B)$ and $T^H X$ are compatible, and therefore

$$\iota^* \operatorname{Todd}(\widehat\nabla^{T^V(\mathbb{S}^N\times B)}) = \operatorname{Todd}(\widehat\nabla^{TB}) \wedge \operatorname{Todd}(\widehat\nabla^{\nu}),$$

which implies that (25) is zero, and therefore $h = 0$. $\square$

One can compare Proposition 2 with [9, Proposition 8.10].

We now prove the dFIT by the flat FIT and the dGRR [10, Theorem 1].

Theorem 4. *Let $\pi : X \to B$ be a proper submersion with closed* spin^c *fibers of even relative dimension. Then*

$$\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}} = \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}} : \widehat K_{\mathrm{FL}}(X) \to \widehat K_{\mathrm{FL}}(B).$$

Proof. Suppose $\mathcal{E}-[n] \in \widehat K_{\mathrm{FL}}(X)$ is a torsion element. Denote by $\mathcal{E}'-[n]' \in K^{-1}_{\mathrm{L}}(X;\mathbb{R}/\mathbb{Z})$ such that $i(\mathcal{E}'-[n]') = \mathcal{E}-[n]$. Then

$$\begin{aligned}
\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E}-[n]) &= \mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(i(\mathcal{E}'-[n]')) \\
&= i(\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}(\mathcal{E}'-[n]')) && \text{by (19)} \\
&= i(\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}(\mathcal{E}'-[n]')) && \text{by the flat FIT (Theorem 3)} \\
&= \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E}-[n]) && \text{by Proposition 2.}
\end{aligned}$$

Now suppose $\mathcal{E} \in \widehat K_{\mathrm{FL}}(X)$ is torsion-free. The dGRR [10, Theorem 1] and [9, Proposition 8.16] (see Proposition 1) imply that $\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E})$ and $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E})$ are torsion-free. Since $\widehat{\mathrm{ch}}_{\mathrm{FL}} : \widehat K_{\mathrm{FL}}(B)\otimes\mathbb{Q} \to \widehat H^{\mathrm{even}}(B;\mathbb{R}/\mathbb{Q})$ is a ring isomorphism, the equality

$$\widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E})) = \widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E})) \tag{26}$$

is equivalent to $\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}(\mathcal{E}) = \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E})$. Note that (26) is a consequence of the dGRR [10, Theorem 1] and [9, Proposition 8.16]. Thus the dFIT is proved in general. $\square$

References

1. M.F. Atiyah and I.M. Singer, *The index of elliptic operators IV*, Ann. of Math. **93** (1971), 119–138.
2. N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004.
3. J.M. Bismut, *The Atiyah-Singer index theorem for families of Dirac operators; two heat equation proofs*, Invent. Math. **83** (1986), 91–151.
4. J.M. Bismut and J. Cheeger, *η-invariants and their adiabatic limits*, J. Amer. Math. Soc. **2** (1989), 33–70.
5. J.M. Bismut and W. Zhang, *Real embeddings and eta invariants*, Math. Ann. **295** (1993), 661–684.
6. U. Bunke and T. Schick, *Smooth K-theory*, Astérisque **328** (2009), 45–135.
7. ———, *Uniqueness of smooth extensions of generalized cohomology theories*, J. Topol. **3** (2010), 110–156.
8. J. Cheeger and J. Simons, *Differential characters and geometric invariants*, in Geometry and Topology (College Park, Md., 1983/84), Lecture Notes in Math. **1167** (1985), 50–80.
9. D. Freed and J. Lott, *An index theorem in differential K-theory*, Geom. Topol. **14** (2010), 903–966.
10. M.H. Ho, *A condensed proof of the differential Grothendieck-Riemann-Roch theorem*, to appear in Proc. Amer. Math. Soc., arXiv:1111.5546v5.
11. M. Hopkins and I.M. Singer, *Quadratic functions in geometry, topology,and M-theory*, J. Diff. Geom. **70** (2005), 329–425.
12. M. Karoubi, *Homologie cyclique et K-théorie*, Astérisque **149** (1987).
13. J. Lott, $\mathbb{R}/\mathbb{Z}$ *index theory*, Comm. Anal. Geom. **2** (1994), 279–311.
14. J. Simons and D. Sullivan, *Structured vector bundles define differential K-theory*, Quanta of Maths, Clay Math. Proc., vol. 11, Amer. Math. Soc., Providence, RI, 2010, pp. 579–599.
15. Z. Yosimura, *Universal coefficient sequences for cohomology theories of CW–spectra*, Osaka J. Math. **12** (1975), 305–323.

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